Journal of Statistical Physics, Vol. 80, Nos. 3/4, 1995

A Functional Relation Among the Pair Correlations of the Two-Dimensional One-Component Plasma

L. Šamaj[1,2] **and J. K. Percus**[1,3]

Received December 13, 1994

We map the classical two-dimensional one-component plasma of charged particles with coupling constant Γ an even positive integer onto a one-dimensional fermionic system. We then show that, in the thermodynamic limit of the fluid regime, translational invariance of the two-body density implies an infinite sequence of interrelations among the coefficients of its short-distance expansion. The existence of these sum rules turns out to be related to a general symmetry of the Coulomb system, providing a functional relation for the two-body density for arbitrary coupling Γ.

KEY WORDS: One-component plasma; logarithmic interaction; sum rules.

1. INTRODUCTION

The classical 2D one-component plasma (OCP) is a system of identical pointlike particles $j=0, 1,..., N-1$ of charge e and position vectors $\mathbf{r}_j$, confined to a disk of radius R whose center is taken as the origin $\mathbf{0}$. The particles are embedded in a spatially uniform neutralizing background of charge density $-en_0$, where $n_0 = N/\pi R^2$ stands for the number density. In two dimensions, the particle-position-dependent part of the Coulomb potential energy Φ of the background–particle system reads[1]

$$\Phi = \sum_j [u(\mathbf{r}_j) + e^2 \pi n_0 r_j^2/2] - e^2 \sum_{j<k} \ln r_{jk} \tag{1}$$

where $r_j = |\mathbf{r}_j|$, $r_{jk} = |\mathbf{r}_j - \mathbf{r}_k|$, and $u(\mathbf{r})$ is an arbitrary external potential; although we will concentrate on the case $u(\mathbf{r}) = 0$ and the thermodynamic

---

[1] Courant Institute of Mathematical Sciences, New York University, New York, New York 10012.
[2] On leave from Institute of Physics, Slovak Academy of Sciences, Bratislava, Slovakia.
[3] Physics Department, New York University, New York, New York 10003.

0022-4715/95/0800-0811$07.50/0

limit $N \to \infty$, it is useful to keep $u(\mathbf{r})$, N as yet undetermined. The corresponding Boltzmann factor

$$\exp(-\beta\Phi) = \prod_j w(\mathbf{r}_j) \prod_{j<k} r_{jk}^{\Gamma} \tag{2}$$

with

$$w(\mathbf{r}) = \exp[-\beta u(\mathbf{r}) - \Gamma\pi n_0 r^2/2] \tag{3}$$

depends on the only dimensionless coupling constant $\Gamma = \beta e^2$. The logarithm of the partition function

$$Z_N = \frac{1}{N!} \int_{r_0<R} d^2\mathbf{r}_0 \cdots \int_{r_{N-1}<R} d^2\mathbf{r}_{N-1} \prod_j w(\mathbf{r}_j) \prod_{j<k} r_{jk}^{\Gamma} \tag{4}$$

is the generating functional for the mean one-particle density

$$n(\mathbf{r}) = \left\langle \sum_j \delta(\mathbf{r}_j - \mathbf{r}) \right\rangle \tag{5}$$

and the two-body density

$$n_2(\mathbf{r}, \mathbf{r}') = \left\langle \sum_{j \neq k} \delta(\mathbf{r}_j - \mathbf{r})\, \delta(\mathbf{r}_k - \mathbf{r}') \right\rangle \tag{6}$$

in the sense that

$$n(\mathbf{r}) = \frac{\delta \ln Z_N}{\delta - \beta u(\mathbf{r})} \tag{7}$$

$$n_2(\mathbf{r}, \mathbf{r}') - n(\mathbf{r})\, n(\mathbf{r}') = w(\mathbf{r}) \frac{\delta n(\mathbf{r})/w(\mathbf{r})}{\delta - \beta u(\mathbf{r}')} \tag{8}$$

Correlations will be considered mainly in the truncated form

$$h(\mathbf{r}, \mathbf{r}') = \frac{n_2(\mathbf{r}, \mathbf{r}') - n(\mathbf{r})\, n(\mathbf{r}')}{n(\mathbf{r})\, n(\mathbf{r}')} \tag{9}$$

Exact results for the thermodynamic and correlation functions are available in the weak coupling $\Gamma \to 0$ limit, as has been shown using the YBG integral equation hierarchy[2] and for the special case $\Gamma = 2$,[3] where the 2D OCP is equivalent to a system of independent fermions. Of particular interest is the Γ dependence of the falloff of the bulk correlations. In the limit $\Gamma \to 0$, they exhibit Debye–Hückel exponential screening, while they are of Gaussian type at $\Gamma=2$. A temperature expansion around $\Gamma=2$[3] indicates the change from a monotonic decay of the two-body density for $\Gamma<2$ to an oscillating one for $\Gamma>2$. The OCP is in a fluid state up to $\Gamma\sim 142$, where it becomes a 2D Wigner crystal.[4] Concerning intermediate values of Γ, it is generally accepted that the pair correlations display a monotonic exponential decay for $0\leqslant\Gamma<2$. In the region $2<\Gamma<142$, estimates for the oscillating correlations range from powerlike falloff,[5] exponentially fast decay in a mean spherical model of the lattice version of plasma,[6] to numerical evidence for Gaussian-type falloff at $\Gamma=4$ coupling.[7]

Besides the limit $\Gamma\to 0$ and the $\Gamma=2$ case (together with the lowest order of the temperature expansion around $\Gamma=2$), charged-fluid sum rules are another important source of exact information about the correlations (for a review see ref. 8). For the 2D OCP, there exist the zeroth-moment (electroneutrality), the second-moment (Stillinger–Lovett),[9,10] and the fourth-moment (compressibility)[11] conditions for the bulk $h(\mathbf{r})$, implied by the specific form of the Coulomb tail at asymptotically large distance.[12] A kind of sum rule is represented also by the Jancovici result[13] relating two lowest-order coefficients of the short-distance expansion of the translational-invariant $h(\mathbf{r})$.

In this paper we show that Jancovici's result[13] represents in fact the lowest level of an infinite sequence of sum rules relating coefficients of the short-distance expansion of $h(\mathbf{r})$. The derivation of these sum rules is outlined in a logical order, with an increasing level of complexity and applicability. We start in Section 2 with the study of a special choice of couplings $\Gamma=2\gamma$, where γ is a positive integer, allowing a Van Der Monde determinantal representation of the Coulomb Boltzmann factor. We map the 2D OCP of N particles onto a 2γ-component fermionic field formulated on a 1D chain of N sites, with spatially inhomogeneous interactions among the fermionic components. In Section 3 we investigate how physical considerations of the homogeneity of the particle density and the translational invariance property of two-body density in the thermodynamic limit $N\to\infty$ manifest themselves in the 1D fermionic system, whose specific correlators determine the spatial dependence of the mentioned quantities. It turns out that the translational invariance of $h(\mathbf{r})$ has, via fermionic correlators, feedback to the coefficients of its short-distance expansion which satisfy an infinite sequence of sum rules. These sum rules imply a functional relation for the pair correlation. It is shown in Section 4 that this functional relation follows from symmetry of the plasma two-body density with respect to a transformation of particle coordinates, valid as well for the most general case of an arbitrary particle number N, inhomogeneous external field $u(\mathbf{r})$, and coupling Γ.

2. THE 2D OCP FORMULATED AS A 1D FERMIONIC SYSTEM

For the sequence of couplings $\Gamma = 2\gamma$ ($\gamma = 1, 2,...$) we can use the Van Der Monde determinantal representation

$$\prod_{j<k} r_{jk}^2 = |\det(r_j e^{i\theta_j})^k|_{j,k=0,...,N-1}|^2 \tag{10}$$

where (r_j, θ_j) are the polar coordinates of $\mathbf{r}_j$, and rewrite the partition function (4) in the form

$$\begin{aligned} Z_N = \frac{1}{N!} \int_{r_0<R} d^2\mathbf{r}_0 \cdots \int_{r_{N-1}<R} d^2\mathbf{r}_{N-1} \prod_j w(\mathbf{r}_j) \det{}_{(1)}(r_j e^{i\theta_j})^k \\ \times \cdots \det{}_{(\gamma)}(r_j e^{i\theta_j})^k \det{}_{(1)}(r_j e^{-i\theta_j})^k \cdots \det{}_{(\gamma)}(r_j e^{-i\theta_j})^k \end{aligned} \tag{11}$$

Every determinant $\det_{(\alpha)}(r_j e^{i\theta_j})^k$, and similarly $\det_{(\alpha)}(r_j e^{-i\theta_j})^k$ ($\alpha = 1,..., \gamma$) can be represented in terms of a set of $2N$ anticommuting variables, say $(\xi_j^{(\alpha)}, \bar{\xi}_j^{(\alpha)})_{j=0}^{N-1}$ and $(\psi_j^{(\alpha)}, \bar{\psi}_j^{(\alpha)})_{j=0}^{N-1}$, satisfying the ordinary Grassmann algebra and the anticommuting integral rules,[14] according to formulas

$$\begin{aligned} &\det{}_{(\alpha)}(r_j e^{i\theta_j})^k|_{j,k=0,...,N-1} \\ &\quad = \int \prod_{j=0}^{N-1} \left\{ d\xi_j^{(\alpha)}\, d\bar{\xi}_j^{(\alpha)} \left[1 + \bar{\xi}_j^{(\alpha)} \sum_{k_\alpha=0}^{N-1} (r_j e^{i\theta_j})^{k_\alpha} \xi_{k_\alpha}^{(\alpha)} \right] \right\} \end{aligned} \tag{12a}$$

$$\begin{aligned} &\det{}_{(\alpha)}(r_j e^{-i\theta_j})^k|_{j,k=0,...,N-1} \\ &\quad = \int \prod_{j=0}^{N-1} \left\{ d\psi_j^{(\alpha)}\, d\bar{\psi}_j^{(\alpha)} \left[1 + \bar{\psi}_j^{(\alpha)} \sum_{l_\alpha=0}^{N-1} (r_j e^{-i\theta_j})^{l_\alpha} \psi_{l_\alpha}^{(\alpha)} \right] \right\} \end{aligned} \tag{12b}$$

The partition function (11) then reads

$$\begin{aligned} Z_N = \frac{1}{N!} \int_{r_0<R} d^2\mathbf{r}_0 \cdots \int_{r_{N-1}<R} d^2\mathbf{r}_{N-1} \prod_j w(\mathbf{r}_j) \\ \times \int \prod_{j=0}^{N-1} \Bigg\{ d\psi_j^{(\gamma)}\, d\bar{\psi}_j^{(\gamma)} \cdots d\psi_j^{(1)}\, d\bar{\psi}_j^{(1)}\, d\xi_j^{(\gamma)}\, d\bar{\xi}_j^{(\gamma)} \cdots d\xi_j^{(1)}\, d\bar{\xi}_j^{(1)} \\ \times \prod_{\alpha=1}^{\gamma} \left[1 + \bar{\xi}_j^{(\alpha)} \sum_{k_\alpha=0}^{N-1} (r_j e^{i\theta_j})^{k_\alpha} \xi_{k_\alpha}^{(\alpha)} \right] \left[1 + \bar{\psi}_j^{(\alpha)} \sum_{l_\alpha=0}^{N-1} (r_j e^{-i\theta_j})^{l_\alpha} \psi_{l_\alpha}^{(\alpha)} \right] \Bigg\} \end{aligned} \tag{13}$$

The determinants induce only bilinear combinations of anticommuting variables "of the same kind" (i.e., with the same index α), so that we can introduce the Grassmann algebra also among "different kinds" of anticommuting variables without changing the value of the anticommuting integral.

Since only combinations which contain all available anticommuting variables contribute to the anticommuting integral, the product over α in (13) for a given particle j can be replaced by

$$\bar{\psi}_j^{(\gamma)} \cdots \bar{\psi}_j^{(1)} \bar{\xi}_j^{(\gamma)} \cdots \bar{\xi}_j^{(1)} \sum_{k_1,\ldots,k_\gamma=0}^{N-1} \sum_{l_1,\ldots,l_\gamma=0}^{N-1} \xi_{k_1}^{(1)} \cdots \xi_{k_\gamma}^{(\gamma)} r_j^{k_1+\cdots+k_\gamma+l_1+\cdots+l_\gamma} \times e^{i\theta_j(k_1+\cdots+k_\gamma-l_1-\cdots-l_\gamma)} \psi_{l_1}^{(1)} \cdots \psi_{l_\gamma}^{(\gamma)} \tag{14}$$

where the anticommutation property of the variables was applied. The resulting function of anticommuting variables becomes diagonal in $(\bar{\xi}, \bar{\psi})$ variables, which integrate out with the trivial factor 1. The polar coordinates of particles do not mix with each other, so we can integrate them separately with the same contribution for every particle:

$$Z_N = \frac{1}{N!} \int \prod_{j=0}^{N-1} d\psi_j^{(\gamma)} \cdots d\psi_j^{(1)}\, d\xi_j^{(\gamma)} \cdots d\xi_j^{(1)} \times \left[\sum_{k_1,\ldots,k_\gamma=0}^{N-1} \sum_{l_1,\ldots,l_\gamma=0}^{N-1} \xi_{k_1}^{(1)} \cdots \xi_{k_\gamma}^{(\gamma)} \times w(k_1+\cdots+k_\gamma, l_1+\cdots+l_\gamma)\, \psi_{l_1}^{(1)} \cdots \psi_{l_\gamma}^{(\gamma)} \right]^N \tag{15}$$

with $w(k, l)$ defined by

$$w(k, l) = \int_{r<R} d^2\mathbf{r}\, w(\mathbf{r})\, r^{k+l} e^{i\theta(k-l)} \tag{16}$$

Equivalently,

$$Z_N = \int \prod_{j=0}^{N-1} d\psi_j^{(\gamma)} \cdots d\psi_j^{(1)}\, d\xi_j^{(\gamma)} \cdots d\xi_j^{(1)} \exp\left[\sum_{k,l=0}^{\gamma(N-1)} \Xi_k w(k, l)\, \Psi_l \right] \tag{17}$$

where the notation

$$\Xi_k = \sum_{\substack{k_1,\ldots,k_\gamma=0 \\ (k_1+\cdots+k_\gamma=k)}}^{N-1} \xi_{k_1}^{(1)} \xi_{k_2}^{(2)} \cdots \xi_{k_\gamma}^{(\gamma)} \tag{18a}$$

$$\Psi_l = \sum_{\substack{l_1,\ldots,l_\gamma=0 \\ (l_1+\cdots+l_\gamma=l)}}^{N-1} \psi_{l_1}^{(1)} \psi_{l_2}^{(2)} \cdots \psi_{l_\gamma}^{(\gamma)} \tag{18b}$$

is used for the combinations of products of γ anticommuting field variables with the given sum of site indices.

To summarize, we have mapped the 2D OCP of N particles to a fermionic system formulated on a 1D chain of N sites. The fermionic field is composed of two, in a certain sense adjoint, sets of anticommuting variables $\{\xi\}$, $\{\psi\}$; each of them contains γ components. The interaction between the fermionic sets is accomplished via the combination (18a) and (18b) of anticommuting variables in the dimensionless Hamiltonian $-\sum_{k,l=0}^{\gamma(N-1)} \Xi_k w(k,l)\, \Psi_l$, where the inhomogeneous coupling $w(k,l)$, defined by (16), reflects the effect of the logarithmic Coulomb interaction as well as the external field. In the special case of an angle-independent potential $u(\mathbf{r})$, $w(r)=w(\mathbf{r})$ implies $w(k,l)=w(k,k)\,\delta_{k,l}$, a partial diagonalization in $\{\Xi, \Psi\}$ variables. For $\gamma=1$, we have the well-known solution $Z_N = \det w(k,l)|_{k,l=0,\ldots,N-1}$.

3. SUM RULES

The fermionic representation of the generator (17) permits us to express formally the spatial dependence of the particle density and of the particle–particle density via the correlators of the 1D fermionic lattice field; in what follows we will use the notation

$$\langle \cdots \rangle = \frac{1}{Z} \int \prod_{j=0}^{N-1} d\psi_j^{(\gamma)} \cdots d\psi_j^{(1)}\, d\xi_j^{(\gamma)} \cdots d\xi_j^{(1)} \times \exp\left[\sum_{k,l=0}^{\gamma(N-1)} \Xi_k w(k,l)\, \Psi_l \right] \cdots \tag{19}$$

for an averaging over all 2γ field components.

The one-particle density (7) is readily obtained in the form

$$n(\mathbf{r}) = w(\mathbf{r}) \sum_{k,l=0}^{\gamma(N-1)} r^{k+l} e^{i\theta(k-l)} \langle \Xi_k \Psi_l \rangle \tag{20}$$

At first it seems that this relation is nothing but a reformulation of the original 2D task to that of the calculation of the 1D fermionic correlators, with a comparable amount of mathematical difficulty. However, Eq. (20) represents an interesting equivalence of the classical and fermionic systems where the fermionic correlators determine the coefficients of the short-distance expansion of $n(\mathbf{r})/w(\mathbf{r})$ and, conversely, physical considerations in two dimensions, such as the homogeneity of the one-particle density of the Coulomb fluid in the absence of an external potential and in the thermodynamic limit, put important restrictions on the fermionic correlators themselves. In particular, for $u(\mathbf{r})=0$ we have $w(\mathbf{r}) = \exp(-\gamma\pi n_0 r^2)$ and, in the limit $N \to \infty$ (which corresponds in the fermionic picture to an infinite number of sites on the 1D chain, with N independence of relevant fermionic correlators), the fermionic coupling (16) reads

$$w(k, l) = \frac{\pi k!}{(\gamma \pi n_0)^{k+1}} \delta_{kl} \tag{21}$$

The resulting diagonalization of the fermionic Hamiltonian in $\{\Xi, \Psi\}$ implies that

$$\langle \Xi_k \Psi_l \rangle = \langle \Xi_k \Psi_k \rangle \delta_{kl} \tag{22}$$

so that

$$n(r) = w(r) \sum_{k=0}^{\infty} \langle \Xi_k \Psi_k \rangle r^{2k} \tag{23}$$

The homogeneity of the particle density, $n(r) = n_0$, then implies

$$\langle \Xi_k \Psi_k \rangle = \frac{(\gamma \pi n_0)^{k+1}}{\pi \gamma k!} \left(= \frac{1}{\gamma w(k, k)} \right) \qquad \text{for all} \quad k = 0, 1, \ldots \tag{24}$$

We see that the assumption of the homogeneous particle density fixes specific fermionic correlators, but this does not provide any new information about the 2D OCP.

On the other hand, the physically motivated invariance of pair correlations in the fluid regime of the 2D OCP, when expressed in terms of the 1D fermionic correlators, feeds back to the internal structure of 2D correlations themselves. To prove this, let us first write down the fermionic formula for the truncated pair correlations (8), (9):

$$\begin{aligned} h(\mathbf{r}, \mathbf{r}') = {} & -1 + \frac{w(\mathbf{r})\, w(\mathbf{r}')}{n(\mathbf{r})\, n(\mathbf{r}')} \\ & \times \sum_{k,l,k',l'=0}^{\gamma(N-1)} r^{k+l} r'^{(k'+l')} e^{i\theta(k-l)} e^{i\theta'(k'-l')} \langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle \end{aligned} \tag{25}$$

For the case of interest $u(\mathbf{r}) = 0$, $N \to \infty$ with the diagonalized form of the fermionic coupling (21),

$$\langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle \neq 0 \qquad \text{iff} \quad k + k' = l + l', \quad k + k' \geqslant \gamma, \quad l + l' \geqslant \gamma \tag{26}$$

The requirement of the equality $k + k' = l + l'$ follows directly from the definition of the averaging over the fermionic fields (19), while the inequalities result from the internal structure of $\{\Xi, \Psi\}$, (18a) and (18b)—as soon as $k + k' < \gamma$ (resp. $l + l' < \gamma$) there exists at least one anticommuting variable

$\xi^{(\alpha)}_{k_\alpha}$ ($\psi^{(\alpha)}_{l_\alpha}$) which occurs twice in the product $\Xi_k \Xi_{k'}$ ($\Psi_l \Psi_{l'}$). In order to simplify the notation, we rescale the fermionic correlators,

$$\langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle = n_0^2 (\gamma \pi n_0)^{(k+l+k'+l')/2} \langle kl \mid k'l' \rangle \tag{27}$$

In the limit $|k-l| \to \infty$, $\langle \Xi_k \Psi_k \Xi_l \Psi_l \rangle$ decouples to $\langle \Xi_k \Psi_k \rangle \langle \Xi_l \Psi_l \rangle$, which, taking advantage of (24), implies that

$$\lim_{|k-l| \to \infty} \langle kk \mid ll \rangle = \frac{1}{k!\, l!} \tag{28}$$

The truncated correlation (25) is now expressed as a function of rescaled distance, defined by

$$\mathbf{x} = (\gamma \pi n_0)^{1/2} \, \mathbf{r} \tag{29}$$

as follows:

$$h(\mathbf{x}, \mathbf{x}') = -1 + e^{-(x^2 + x'^2)} \times \sum_{\substack{k,l,k',l'=0 \\ (\{k+k'=l+l'\} \geqslant \gamma)}}^{\gamma(N-1)} x^{k+l} x'^{(k'+l')} e^{i\theta(k-l)} e^{i\theta'(k'-l')} \langle kl \mid k'l' \rangle \tag{30}$$

To establish a convenient format for incorporating the translational invariance property $h(\mathbf{x}, \mathbf{x}') = h(|\mathbf{x} - \mathbf{x}'|)$, we first consider the explicit form of the truncated correlation between particles localized at the origin $\mathbf{x}' = \mathbf{0}$ and at an arbitrary point $\mathbf{x}$,

$$h(x) = -1 + e^{-x^2} \sum_{k=\gamma}^{\infty} \langle kk \mid 00 \rangle \, x^{2k} \tag{31}$$

The coefficients $\{\langle kk \mid 00 \rangle\}$ are positive numbers because in the averaging over anticommuting fields with the Hamiltonian diagonal in $\{\Xi, \Psi\}$ the parity of a permutation of anticommuting variables building Ξ_k is exactly the same as that of the building elements of Ψ_k. The asymptotic values

$$\lim_{k \to \infty} \langle kk \mid 00 \rangle = 1/k! \tag{32}$$

follow from (28). Substituting then

$$x^2 \to |\mathbf{x} - \mathbf{x}'|^2 = (xe^{i\theta} - x'e^{i\theta'})(xe^{-i\theta} - x'e^{-i\theta'})$$

in (31) and comparing with (30), we get

$$\binom{k}{r}\binom{k}{s}\langle kk \mid 00\rangle$$
$$= \sum_{l=\max(0,r+s-k)}^{r} \sum_{m=\max(0,r+s-k)}^{s} (-1)^{l+m}$$
$$\times \frac{1}{(r-l)!\,(s-m)!}\langle lm \mid k+m-r-s, k+l-r-s\rangle \tag{33}$$

where r, $s=0, 1,..., k$, and this can be inverted to read

$$\langle kl \mid k'l'\rangle = \sum_{r=\max(0,k-l')}^{k} \sum_{s=\max(0,l-k')}^{l} (-1)^{r+s} \frac{1}{(k-r)!\,(l-s)!}$$
$$\times \binom{l'-k+r+s}{r}\binom{k'-l+r+s}{s}$$
$$\times \langle l'-k+r+s, k'-l+r+s \mid 00\rangle \tag{34}$$

under the constraint $k+k'=l+l'$. Some of the interrelations among the correlators can be readily verified by using the anticommutation rules for the building elements of $\{\Xi, \Psi\}$ together with the evident interchange symmetries among the fermionic field components. The other ones are non-trivial consequences of the translational invariance of $h(\mathbf{x}, \mathbf{x}')$ and cannot be derived "directly" from the microscopic fermion model. A special case of these interrelations is represented by the choice $r=k$, $s=0$ in (33) [or $l=0$, $k'=0$ in (34)]:

$$\langle kk \mid 00\rangle = \sum_{l=\gamma}^{k} (-1)^{l} \frac{1}{(k-l)!}\langle l0 \mid 0l\rangle \tag{35}$$

Since $\langle l0 \mid 0l\rangle = \langle \Xi_l \Psi_0 \Xi_0 \Psi_l\rangle = \langle \Xi_l \Psi_l \Psi_0 \Xi_0\rangle$, while $\Psi_0 \Xi_0 = (-1)^{\gamma}\, \Xi_0 \Psi_0$, we see that $\langle l0 \mid 0l\rangle = (-1)^{\gamma}\langle ll \mid 00\rangle$. Finally, then,

$$\langle kk \mid 00\rangle = \sum_{l=\gamma}^{k} (-1)^{\gamma+l} \frac{1}{(k-l)!}\langle ll \mid 00\rangle \qquad \text{for} \quad k=\gamma, \gamma+1,... \tag{36}$$

The structure of the infinite set of linear relations (36) among the coefficients of the $h(x)$ expansion (31) is the following. When $k=\gamma+2m$

$(m=1,2,\ldots)$, $\langle kk \mid 00\rangle$ on the lhs of (36) cancels with its counterpart on the rhs, and we have

$$\langle \gamma+2m-1, \gamma+2m-1 \mid 00\rangle = \sum_{l=0}^{2(m-1)} (-1)^l \frac{1}{(2m-l)!} \langle \gamma+l, \gamma+l \mid 00\rangle \tag{37}$$

When $k=\gamma+2m-1$ $(m=1,2,\ldots)$, we get directly

$$\begin{aligned} &\langle \gamma+2m-1, \gamma+2m-1 \mid 00\rangle \\ &\quad = \frac{1}{2} \sum_{l=0}^{2(m-1)} (-1)^l \frac{1}{(2m-l-1)!} \langle \gamma+l, \gamma+l \mid 00\rangle \end{aligned} \tag{38}$$

However, as we will see in (42), the recursions (37), (38) generate the *same* sequence of $\{\langle \gamma+2m-1, \gamma+2m-1 \mid 00\rangle\}$ expressed in terms of lower-order coefficients from the set $\{\langle \gamma+2m, \gamma+2m \mid 00\rangle\}$; for $m=1$ one finds

$$\langle \gamma+1, \gamma+1 \mid 00\rangle = \frac{1}{2!} \langle \gamma\gamma \mid 00\rangle \tag{39}$$

which is nothing but Jancovici's result[13] adapted from the original derivation for three dimensions to 2D jellium; for $m=2$ we have

$$\langle \gamma+3, \gamma+3 \mid 00\rangle = \frac{1}{2!} \langle \gamma+2, \gamma+2 \mid 00\rangle - \frac{1}{4!} \langle \gamma\gamma \mid 00\rangle \tag{40}$$

and so on.

The two-body density

$$n_2(x) = n_0^2 e^{-x^2} \sum_{k=\gamma}^{\infty} \langle kk \mid 00\rangle \, x^{2k} \tag{41}$$

is, up to the prefactor, the generating function for the set of coefficients $\{\langle kk \mid 00\rangle\}$. Multiplying both sides of (36) by x^{2k} and summing over all $k=\gamma, \gamma+1,\ldots$, we obtain the functional relation for $n_2(x)$,

$$n_2(x) = (-1)^\gamma e^{-x^2} n_2(ix) \tag{42}$$

which, when written as

$$e^{x^2/2} n_2(x) = (-1)^\gamma e^{(ix)^2/2} n_2(ix)$$

gives no information on half of the powers of x in the power series expansion of $\exp(x^2/2)\; n_2(x)$; thus we see that one-half of the infinite sequence of sum rules (36) is ineffective. As a consequence of (42),

$$\begin{aligned} h(x) &= -1 + n_2(x)/n_0^2 \\ &= -1 + (-1)^{\gamma}\, e^{-x^2} n_2(ix)/n_0^2 \\ &= -1 + \sum_{k=\gamma}^{\infty} (-1)^{\gamma+k} \langle kk \mid 00 \rangle\, x^{2k} \end{aligned} \tag{43}$$

i.e., the positive coefficients $\{\langle kk \mid 00\rangle\}$, with alternating $+$, $-$ signs attached, are the true coefficients of the short-distance expansion of the truncated pair correlation. The functional relation for $n_2(x)$, (42), also restricts $h(x)$ to the form

$$h(x) = -1 + e^{-x^2/2} x^{2\gamma} H(x^4) \tag{44}$$

with H a MacLaurin series in its argument. The asymptotic behavior of $h(x)$, $\lim_{x\to\infty} h(x) = 0$, yields

$$\lim_{x\to\infty} H(x^4) \sim e^{x^2/2} x^{-2\gamma} \tag{45}$$

In particular, in the well-known case $\gamma = 1$, one has the exact result $H(x^4) = (\sinh x^2/2)/(x^2/2)$.

4. GENERALIZATION

The transformation formula (42) is valid only in the thermodynamic limit and for zero external field, when the two-body correlation becomes translational invariant. To understand more deeply the origin of the symmetry the formula comes from, it would be useful to know the equivalent of the functional relation (42) in the most general case of finite N and arbitrary inhomogeneity of the external potential, from which (42) would follow as a trivial consequence of translational invariance. This task is accomplished by a series of straightforward transformations for the ratio $n_2(z_1, z_1^*; z_2, z_2^*)/[w(z_1, z_1^*)\, w(z_2, z_2^*)]$, expressed by using the fermionic representation of the truncated correlation (25) and the complex number notation $z = r\exp(i\theta)$:

$$\begin{aligned}
&\frac{n_2(z_1, z_1^*; z_2, z_2^*)}{w(z_1, z_1^*)\, w(z_2, z_2^*)} \\
&\quad = \sum_{k,l,k',l'=0}^{\gamma(N-1)} z_1^k z_1^{*l} z_2^{k'} z_2^{*l'} \langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle \\
&\quad = (-1)^\gamma \sum_{k,l,k',l'=0}^{\gamma(N-1)} z_1^k z_1^{*l} z_2^{k'} z_2^{*l'} \langle \Xi_{k'} \Psi_l \Xi_k \Psi_{l'} \rangle \\
&\quad = (-1)^\gamma \sum_{k,l,k',l'=0}^{\gamma(N-1)} z_2^k z_1^{*l} z_1^{k'} z_2^{*l'} \langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle \\
&\quad = (-1)^\gamma \frac{n_2(z_2, z_1^*; z_1, z_2^*)}{w(z_2, z_1^*)\, w(z_1, z_2^*)}
\end{aligned} \tag{46}$$

In the limit $N \to \infty$ and under homogeneous external conditions $w(z, z^*) = \exp(-\gamma\pi\rho z z^*)$, $n_2(z_1, z_1^*; z_2, z_2^*)$ depends only on the distance $[(z_1 - z_2)(z_1^* - z_2^*)]^{1/2}$ and we arrive at

$$\frac{n_2([(z_1 - z_2)(z_1^* - z_2^*)]^{1/2})}{\exp[-\gamma\pi\rho(z_1 z_1^* + z_2 z_2^*)]} = (-1)^\gamma \frac{n_2(i[(z_1 - z_2)(z_1^* - z_2^*)]^{1/2})}{\exp[-\gamma\pi\rho(z_2 z_1^* + z_1 z_2^*)]} \tag{47}$$

Expressed in terms of the dimensionless distance x, Eq. (47) is indeed equivalent to (42), as was required.

We see that the transformation of coordinates which leaves [up to the prefactor $(-1)^\gamma$] the ratio $n_2(z_1, z_1^*; z_2, z_2^*)/[w(z_1, z_1^*)\, w(z_2, z_2^*)]$ invariant reads

$$z_1' = z_2, \qquad z_1'^* = z_1^*, \qquad z_2' = z_1, \qquad z_2'^* = z_2^* \tag{48}$$

Written in the center-of-mass vector basis

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{49}$$

it takes a more transparent form

$$\mathbf{R}' = \mathbf{R}, \qquad \mathbf{r}' = i\hat{\mathbf{z}} \times \mathbf{r} \tag{50}$$

where $\hat{\mathbf{z}}$ is the unit vector in the z direction perpendicular to the (x, y) plane.

We are now ready to extend the treatment to the case of general $\Gamma = 2\gamma$ ($\gamma \in R^+$). Let us write down the explicit formula for the pair correlation $n_2(\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)$, (6), divided by all Boltzmann factors which are, due to the averaging of δ-functions, dependent only on the particle vectors $\mathbf{r}_1$, $\mathbf{r}_2$, or on both of them:

$$\frac{n_2(\mathbf{r}_1, \mathbf{r}_2)}{w(\mathbf{r}_1)\, w(\mathbf{r}_2) \exp(\gamma \ln r_{12}^2)} = \frac{N(N-1)}{2Z_N} \int_{r_3 < R} d^2\mathbf{r}_3 \cdots \int_{r_N < R} d^2\mathbf{r}_N \times \prod_3^N w(\mathbf{r}_j) \prod_{2<j<k<N} e^{\gamma \ln r_{jk}^2} \prod_3^N e^{\gamma \ln (r_{1j}^2 r_{2j}^2)} \tag{51}$$

The product $(z_1 - z_j)(z_1^* - z_j^*)(z_2 - z_j)(z_2^* - z_j^*)$ is invariant with respect to the transformation of coordinates (48) for *every* $j = 3, \ldots, N$. Consequently, the transformation (48) leaves the whole integral over $\mathbf{r}_3, \ldots, \mathbf{r}_N$ unchanged, which results in the equivalence

$$\frac{n_2(z_1, z_1^*; z_2, z_2^*)}{w(z_1, z_1^*)\, w(z_2, z_2^*)[(z_1 - z_2)(z_1^* - z_2^*)]^\gamma} = \frac{n_2(z_2, z_1^*; z_1, z_2^*)}{w(z_2, z_1^*)\, w(z_1, z_2^*)[(z_2 - z_1)(z_1^* - z_2^*)]^\gamma} \tag{52}$$

In the regime of translationally invariant correlations, with the dimensionless particle–particle distance x defined by $x^2 = \gamma\pi\rho(z_1 - z_2)(z_1^* - z_2^*)$, we find

$$n_2(x) = (-1)^\gamma e^{-x^2} n_2(ix) \tag{53}$$

which now holds for arbitrary $\gamma \geqslant 0$ in the fluid-phase regime, when the appropriate branch of $(-1)^\gamma$ is chosen.

5. CONCLUDING REMARKS

The main result of the present work consists in the invariance of the ratio $n_2(\mathbf{r}_1, \mathbf{r}_2)/[w(\mathbf{r}_1)\, w(\mathbf{r}_2) \exp(\gamma \ln r_{12}^2)]$ under the transformation (49), (50) of particle vectors $\mathbf{r}_1$, $\mathbf{r}_2$, which implies the simple functional formula (53) for the translation-invariant two-body density of the 2D OCP fluid.

It is easy to show that the established symmetry uniquely defines the logarithmic potential, but we do not know whether a representation in which all symmetries of the model system are realized automatically is sufficient for solving its thermodynamics exactly, "at least" for γ a positive integer when the short-distance expansion of the two-body density is analytical. It stands to reason that the treatment can be straightforwardly extended to all higher-order correlations: their symmetry with respect to the transformation (49), (50) of coordinates evidently applies for every couple of particle indices. If there are no other symmetries mixing the coordinates of three, four,... particles (which is questionable), the model might be solvable using the infinite hierarchies of YBG or linear response[15] type.

Another possible way to attack the problem is to concentrate on the microscopic 1D fermionic model for the set of integer γ's. This model, besides being the primary tool for finding our new symmetry property of jellium, provides additional exact information, e.g., it helped us to establish the positivity of the coefficients $\{\langle kk \mid 00 \rangle\}$ of the short-distance expansion of $h(x)$ and to determine their asymptotic values (32). The simple form of the correlators $\langle \Xi_k \Psi_k \rangle$, (24), and the possibility to express the whole set of correlators $\langle \Xi_k \Psi_l \Xi_{k'} \Psi_{l'} \rangle$ using one particular sequence $\{\langle \Xi_k \Psi_k \Xi_0 \Psi_0 \rangle\}$ [see (34)] indicate that there might be some relevant simplification in the $\{\Xi, \Psi\}$ algebra in the limit of an infinite number of chain sites.

The general validity of the transformation formula (52) suggests its potential application to the inhomogeneous crystal phase of jellium, too. It is also worthwhile to search for a symmetry of the kind presented for Coulomb systems in three and higher spatial dimensions.

ACKNOWLEDGMENT

This work was supported by a grant from the National Science Foundation.

REFERENCES

1. B. Jancovici, *Inhomogeneous Fluids*, D. Henderson, ed. (Dekker, New York, 1992), pp. 201–237.
2. L. Blum, C. Gruber, J. L. Lebowitz, and P. Martin, *Phys. Rev. Lett.* **48**:1769 (1982).
3. B. Jancovici, *Phys. Rev. Lett.* **46**:386 (1981).
4. Ph. Choquard and J. Clérouin, *Phys. Rev. Lett.* **50**:2086 (1983).
5. J. Piasecki and D. Levesque, *J. Stat. Phys.* **47**:489 (1987).
6. E. R. Smith, *J. Stat. Phys.* **50**:813 (1988).
7. L. Šamaj, J. K. Percus, and M. Kolesík, *Phys. Rev. E* **49**:5623 (1994).
8. Ph. A. Martin, *Rev. Mod. Phys.* **60**:1075 (1988).
9. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **48**:3858 (1968).
10. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **49**:1991 (1968).
11. P. Vieillefosse and J. P. Hansen, *Phys. Rev. A* **12**:1106 (1975).
12. Ph. Attard, *Phys. Rev. E* **48**:3604 (1993).
13. B. Jancovici, *J. Stat. Phys.* **17**:357 (1977).
14. F. A. Berezin, *The Method of Second Quantization* (Academic Press, New York, 1966).
15. M. S. Wertheim, *J. Chem. Phys.* **65**:2377 (1976).